Tits p-indexes of semisimple algebraic groups

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Abstract

The first author has recently shown that semisimple algebraic groups are classified up to motivic equivalence by the local versions of the classical Tits indexes over field extensions, known as Tits p-indexes. We provide in this article the complete description of the values of the Tits p-indexes over fields. From this exhaustive study, we also deduce criteria for motivic equivalence of semisimple groups of many types, hence giving a dictionary between classic algebraic structures, representation theory, cohomological invariants and Chow motives of the twisted flag varieties for those groups.

The *Tits index* (sometimes called Satake diagram) of a semisimple linear algebraic group G over a field k includes as special cases the classical notions of Schur index of a central simple associative algebra and the Witt index of a quadratic form. It is a fundamental invariant of semisimple algebraic groups. However, for the purpose of stating and proving theorems about Chow motives with $\mathbb{F}_p$ coefficients, one should consider not the Tits index of G, but rather the (Tits) p-index, meaning the Tits index of G_L where L is an algebraic extension of k of degree not divisible by p, yet all the finite algebraic extensions of L have degree a power of p. Such an L is called a *p-special closure* of k in [**9**, §101.B] and all such fields are isomorphic as k-algebras, so the notion of Tits p-index over k is well defined.

Let G be a semisimple algebraic group over k. As shown in [**8**], the Tits p-indexes of G on all fields extensions of k — the *higher Tits p-indexes of G* — determine the motivic equivalence class of G modulo p. The aim of this article is to determine the values of the Tits p-indexes of the absolutely simple algebraic groups, using as a starting point the known list of possible Tits indexes as in [**47**], [**45**], or [**38**]. Along the way, we give in some cases criteria for motivic equivalence for semisimple groups in terms of their algebraic and cohomological invariants.

1. Generalities

1.1. *Definition of the Tits index.*

References for this subsection: [**47**], [**38**], [**30**, §1]

The *Tits index* is (1) the Dynkin diagram of G, which we conflate with its set Δ of vertices, together with (2) the action of the absolute Galois group $\mathrm{Gal}(k)$ of k on Δ, and (3) a $\mathrm{Gal}(k)$-invariant subset $\Delta_0 \subset \Delta$. Specifically, pick a maximal k-torus T in G containing a maximal k-split torus S. For k_{sep} a separable closure of k — so $\mathrm{Gal}(k)$ are the k-automorphisms of k_{sep} — $T \times k_{\mathrm{sep}}$ and $G \times k_{\mathrm{sep}}$ are split, and, from the set Φ of roots of $G \times k_{\mathrm{sep}}$ with respect to $T \times k_{\mathrm{sep}}$ one picks a set of simple roots Δ. As T is k-defined, $\mathrm{Gal}(k)$ acts naturally on Φ;

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this action need not preserve Δ, but modifying it by elements of the Weyl group in a natural way gives a canonical action of $\mathrm{Gal}(k)$ on Δ, called the $*$-*action*. The resulting graph (Dynkin diagram with vertex set Δ) with action by $\mathrm{Gal}(k)$ is uniquely defined up to isomorphism in the category of graphs with a $\mathrm{Gal}(k)$-action; it does not depend on the choice of T or Δ.

This addresses parts (1) and (2) of the Tits index. To define Δ_0 in (3), choose orderings on T^* (equivalently, on Φ, in the sense of [**5**, §VI.1.6]) and S^* such that the linear map, restriction $T^* \to S^*$, takes nonnegative elements of T^* to nonnegative elements of S^*. Take Δ to be the set of simple roots in Φ with respect to this ordering. Define Δ_0 to be the set of $\alpha \in \Delta$ such that $\alpha|_S = 0$; it is a union of $\mathrm{Gal}(k)$-orbits in Δ. One has $\Delta_0 = \Delta$ iff G is anisotropic and $\Delta_0 = \emptyset$ iff G is quasi-split. The elements of $\delta_0 = \Delta \setminus \Delta_0$ are called *distinguished* and the number of $\mathrm{Gal}(k)$-orbits of distinguished elements equals the rank of a maximal k-split torus in G.

To represent the Tits index graphically, one draws the Dynkin diagram and circles the distinguished vertices. Traditionally, one indicates the Galois action by drawing vertices in the same $\mathrm{Gal}(k)$-orbit physically close to each other on the page, and by using one large circle or oval to enclose each $\mathrm{Gal}(k)$-orbit in δ_0. The Tits index of G has no circles iff G is anisotropic, and every vertex is circled iff G is quasi-split.

The definition of Tits index is compatible with base change, as explained carefully in [**39**, pp. 115, 116]. That is, for each extension E of k, the Tits index of $G \times E$ may be taken to have the same underlying graph (the Dynkin diagram with vertex set Δ) with $\mathrm{Gal}(E)$-action given by the restriction map $\mathrm{Gal}(E) \to \mathrm{Gal}(k)$, and with set of distinguished vertices containing the distinguished vertices in the Tits index of G. It follows that the Tits p-index of G, as defined in the first paragraph of this paper, is also compatible with base change.

1.2. Which primes p?

If every simple group of a given type is split by a separable extension of degree not divisible by p, then the only possible p-index is the split one. Thus, [**43**, §2.2] gives a complete list $S(G)$ of the primes meriting consideration, which we reproduce in Table 1. (Or see [**49**] for more precise information on degrees of splitting fields.)

To say this in a different way, we fix a prime p and will describe the possible Tits indexes of a simple algebraic group over a field k that is p-*special*, i.e., such that every finite extension has degree a power of p. A minimal p-special extension of k, i.e., a p-special closure of k, can be constructed as follows. If k is perfect or has characteristic p, take the subfield of k_{sep} fixed by a p-Sylow subgroup of $\mathrm{Gal}(k)$. If $\mathrm{char}\, k$ is positive but different from p, perform the same construction but on the perfect closure of k.

type of G	exponent of the center	elements of $S(G)$
A_n	$n+1$	2 and the prime divisors of $n+1$
$B_n, C_n, D_n\ (n \neq 4)$	2	2
G_2	1	2
D_4, E_7	2	2 and 3
F_4	1	2 and 3
E_6	3	2 and 3
E_8	1	2, 3, and 5

TABLE 1. *Primes $S(G)$*

1.3. Twisted flag varieties and motivic equivalence.

References for this subsection: [**4**], [**3**], [**30**, §1]

For each subset Θ of Δ, there is a parabolic subgroup P_Θ of $G \times k_{\mathrm{sep}}$ (determined by the choice of simple roots) whose Levi subgroup has Dynkin diagram $\Delta \setminus \Theta$. The (projective) quotient variety $(G \times k_{\mathrm{sep}})/P_\Theta$ is the variety of parabolic subgroups of $G \times k_{\mathrm{sep}}$ that are conjugate to P_Θ. This variety is k-defined if and only if Θ is invariant under $\mathrm{Gal}(k)$, in which case we denote it by X_Θ. These varieties are the *twisted flag varieties* of G.

Now fix a $\mathrm{Gal}(k)$-invariant subset Θ of Δ. The following are equivalent: *(1)* X_Θ *has a k-point;* *(2)* X_Θ *is a rational variety;* *(3)* $\Theta \subseteq \delta_0$. In this way, the Tits index of G gives information about X_Θ.

In case X_Θ does not have a k-point, the Chow motive of X_Θ nonetheless gives information about the geometry of X_Θ. Suppose now that G' is a semisimple group over k, and that the quasi-split inner forms G_{ν_G}, $G'_{\nu_{G'}}$ of G, G' are isogenous. That is, suppose that there is an isomorphism f from the Dynkin diagram Δ of G to that of G' that commutes with the action of $\mathrm{Gal}(k)$. This defines a correspondence $X_\Theta \leftrightarrow X_{f(\Theta)}$ between the twisted flag varieties of G and G'. The groups G, G' are *motivic equivalent modulo a prime p* if there is a choice of f such that the mod-p Chow motives of X_Θ and $X_{f(\Theta)}$ are isomorphic for every $\mathrm{Gal}(k)$-stable $\Theta \subseteq \Delta$. The groups G, G' are *motivic equivalent* if the groups are motivic equivalent mod p for every prime p (where f may depend on p). The main result of [**8**] says, for a prime p: G and G' are motivic equivalent mod p iff there is an isomorphism f whose base change to each p-special field E containing k identifies the distinguished vertices in the Tits index of $G \times E$ with those in the Tits index of $G' \times E$. Informally, G and G' are motivic equivalent mod p iff $G \times K$ and $G' \times K$ have the same Tits p-index for every extension K of k. This theorem is one motivation for our study of the possible Tits p-indexes of semisimple groups.

1.4. The quasi-split type of G.

A group G over k has *quasi-split type* ${}^t T_n$ if, upon base change to a separable closure k_{sep} of k, $G \times k_{\mathrm{sep}}$ is split with root system of type T_n and if the image of $\mathrm{Gal}(k) \to \mathrm{Aut}(\Delta)$ has order t. If $t = 1$, then G is said to have *inner type* and otherwise G has *outer type*. In the case where k is p-special, evidently t must be a power of p.

1.5. The Tits class of G.

Suppose that G is adjoint with simply connected cover $\widetilde{G}$. One has an exact sequence $1 \to Z \to \widetilde{G} \to G \to 1$ where Z is the scheme-theoretic center of $\widetilde{G}$. This gives a connecting homomorphism $\partial \colon H^1(k, G) \to H^2(k, Z)$; here and below H^i denotes fppf cohomology. There is a unique class $\nu_G \in H^1(k, G)$ such that twisting G by ν_G gives a quasi-split group [**25**, 31.6], and we call $t_G := -\partial(\nu_G) \in H^2(k, Z)$ the *Tits class* of G.

As a finite abelian group scheme, there is a unique minimal natural number n such that multiplication by n is the zero map on Z, it is called the exponent of Z. If p does not divide n and k is p-special, then the Tits class of G is necessarily zero.

LEMMA 1.   *Suppose G is a semisimple adjoint algebraic group with simply connected cover $\widetilde{G}$. If $t_G = 0$, then there is a unique class $\xi_G \in H^1(k, \widetilde{G})$ so that the twisted group $\widetilde{G}_{\xi_G}$ is quasi-split.*

Proof.   As $t_G = 0$, the exactness of the sequence $H^1(k, \widetilde{G}) \to H^1(k, G) \xrightarrow{\partial} H^2(k, Z)$ shows that there is a $\xi \in H^1(k, \widetilde{G})$ mapping to ν_G, and it remains to prove uniqueness.

By twisting, it is the same to show that, for $\widetilde{G}$ quasi-split simply connected, the map $H^1(k, \widetilde{G}) \to H^1(k, \mathrm{Aut}(\widetilde{G}))$ has zero kernel. By [**17**, Theorem 11, Example 15], the kernel of the map is the image of $H^1(k, Z) \to H^1(k, \widetilde{G})$. As Z is contained in every maximal torus of

$\widetilde{G}$, at least one of which, call it S, is quasi-trivial, the map factors through $H^1(k, S)$, which is zero by Hilbert's Theorem 90. $\qquad\square$

1.6. The Tits algebras of G.

References for this subsection: [**25**, §27], [**48**]

Let $\widetilde{G}$, Z be as in the previous subsection. The Tits class provides, by way of the Tits algebras, a cohomological obstruction for an irreducible representation of $\widetilde{G} \times k_{\text{sep}}$ over k_{sep} to be defined over k. Specifically, such a representation has highest weight a dominant weight λ. Put $k(\lambda)$ for the subfield of k_{sep} of elements fixed by the stabilizer of λ in $\operatorname{Gal}(k)$ (under the $*$-action). The weight λ is fixed by $\operatorname{Gal}(k(\lambda))$, i.e., λ restricts to a homomorphism $Z \to \mathbb{G}_m$. The *Tits algebra* $A_G(\lambda)$ is the image of t_G under the induced map $\lambda \colon H^2(k(\lambda), Z) \to H^2(k(\lambda), \mathbb{G}_m)$; the irreducible representation is defined over $k(\lambda)$ iff $A_G(\lambda) = 0$. (Note that in this definition $A_G(\lambda)$ is only a Brauer class, but a more careful definition gives a central simple algebra whose degree equals the dimension of the representation.)

As $H^2(k, Z)$ is a torsion abelian group where every element has order dividing the exponent of Z, there is a unique element $t_{G,p}$ of p-primary order such that $t_G - t_{G,p}$ has order not divisible by p. We write $A_{G,p}(\lambda) \in H^2(k(\lambda), \mathbb{G}_m)$ for the image of $t_{G,p}$ under λ; it is the p-primary component of the Brauer class $A_G(\lambda)$. We now show that if G and G' are motivic equivalent mod p, then they must have the same Tits algebras "up to prime-to-p extensions". Note that if G and G' are motivic equivalent mod p for some p, then the isomorphism f provides an isomorphism $f^* \colon Z' \to Z$ of the centers of their simply connected covers.

PROPOSITION 2. *Suppose G and G' are absolutely simple algebraic groups that are motivic equivalent mod p via an isomorphism of Dynkin diagrams f. Then $t_{G,p}$ and $f^*(t_{G',p})$ generate the same subgroup of $H^2(k, Z)$ and, for every dominant weight λ, $A_{G,p}(\lambda)$ and $A_{G',p}(f(\lambda))$ generate the same subgroup of $H^2(k(\lambda), \mathbb{G}_m)$.*

Proof. The claim is trivial unless G has type A, B, C, D, E_6, or E_7 and p divides the exponent of Z. (We remark that $t_{G,p} = t_G$ except possibly when G has type A.)

Suppose first that G has inner type. We verify the claim about $A_{G,p}(\lambda)$ and $A_{G',p}(f(\lambda))$ for a given dominant weight λ that we may assume is not in the root lattice. Then there is a unique minuscule weight λ_0 congruent to λ module the root lattice, and we have $A_{G,p}(\lambda) = A_{G,p}(\lambda_0)$ and similarly for G'. Replacing λ with λ_0, we may assume that λ is a fundamental dominant weight and therefore is dual to a simple coroot $\alpha^\vee$. Set K to be a p-special closure of the function field of the twisted flag variety X_α for G. Then α is distinguished in the Tits index of $G \times K$ (trivially), hence $f(\alpha)$ is distinguished for $G' \times K$ and $\operatorname{res}_{K/k} A_{G',p}(f(\lambda))$ is zero [**48**, p. 211]. But the kernel of $\operatorname{res}_{K/k}$, on the p-primary part of $H^2(k, \mathbb{G}_m)$, is generated $A_{G,p}(\lambda)$ [**32**, Th. B]. By symmetry $A_{G,p}(\lambda)$ and $A_{G',p}(f(\lambda))$ generate the same subgroup of the p-primary part of the Brauer group. If G has type other than D_n for even $n \geq 4$, then the character group Z^* is cyclic; taking λ to be a generator we find an m not divisible by p so that $mA_{G,p}(\lambda) = A_{G,p'}(f(\lambda))$. In the excluded case, Z has exponent 2 and (assuming $p = 2$) $A_{G,2}(\lambda) = A_{G',2}(f(\lambda))$ for all $\lambda \in Z^*$ and we set $m = 1$. Thus, since the map $H^2(k, Z) \to \prod_\lambda H^2(k(\lambda), Z)$ is injective [**17**, Prop. 7], we find that $mt_{G,p} = f^*(t_{G',p})$.

Now suppose that G has outer type. We may replace k by a p-special closure and so suppose that G becomes inner type over an extension of degree p and that p divides the exponent of Z (for otherwise $t_{G,p}$ and $t_{G',p}$ are necessarily zero). Thus we may assume that $p = 2$ and G has type $^2A_{n-1}$ with even $n \geq 4$ or type ^2D_n for $n \geq 4$. We give the details for type A; the type D case is easier.

Write $G = \mathrm{SU}(B, \tau)$ and $G' = \mathrm{SU}(B', \tau')$ for B, B' central simple K-algebras of degree n for some separable quadratic extension K of k, and τ, τ' K/k-involutions on B, B' respectively as in [25, pp. 366, 367]. The fundamental weight $\lambda_{n/2}$ in the center of the Dynkin diagram (Table 2) is fixed by $\mathrm{Gal}(k)$ and has Tits algebra $A_{G,2}(\lambda_{n/2})$ the class of the discriminant algebra $D(B, \tau)$ of (B, τ) as in [25, p. 378] and similarly for G'. The argument as for G of inner type shows that $D(B, \tau)$ and $D(B', \tau')$ have the same class in the Brauer group $H^2(k, \mathbb{G}_m)$. Replacing k by K and applying the result for the inner case shows that B and B' generate the same subgroup (necessarily of order a power of 2) in $H^2(K, \mathbb{G}_m)$. Therefore, there is an odd number m such that $mA_{G,2}(\lambda_1) = A_{G',2}(f(\lambda_1))$ for λ_1 the fundamental weight corresponding to the far left vertex in the Dynkin diagram, hence, for λ_i the fundamental weight dual to the simple coroot $\alpha_i^\vee$, we have $mA_{G,2}(\lambda_i) = miA_{G,2}(\lambda_1) = iA_{G',2}(f(\lambda_1)) = A_{G',2}(f(\lambda_i))$ in $H^2(K, \mathbb{G}_m)$. As of course $mA_{G,2}(\lambda_{n/2}) = A_{G,2}(\lambda_{n/2})$, we find that $mt_{G,2}$ and $f^*(t_{G',2})$ have the same image under the injective map $H^2(k, Z) \to \prod_\lambda H^2(k(\lambda), Z)$ hence are equal. $\qquad\square$

1.7. The Rost invariant.

References for this subsection: [20], [25]

We refer to [20, pp. 105–158] for the precise definition of the abelian torsion groups $H^3(k, \mathbb{Z}/d\mathbb{Z}(2)) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$; if d is not divisible by $\mathrm{char}\, k$, then $H^3(k, \mathbb{Z}/d\mathbb{Z}(2)) = H^3(k, \mu_d^{\otimes 2})$. For all k, the natural inclusion identifies $H^3(k, \mathbb{Z}/d\mathbb{Z}(2))$ with the d-torsion in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$. For $\widetilde{G}$ a simple simply connected algebraic group, there is a canonical morphism of functors

$$r_{\widetilde{G}} \colon H^1(*, \widetilde{G}) \to H^3(*, \mathbb{Q}/\mathbb{Z}(2))$$

known as the *Rost invariant*. The order $n_{\widetilde{G}}$ of $r_{\widetilde{G}}$ is known as the Dynkin index of $\widetilde{G}$, and $r_{\widetilde{G}}$ can be viewed as a morphism $H^1(*, \widetilde{G}) \to H^3(*, \mathbb{Z}/n_{\widetilde{G}}\mathbb{Z}(2))$.

LEMMA 3. *Let $\widetilde{G}$ be absolutely simple and simply connected with center Z. Put m for the largest divisor of $n_{\widetilde{G}}$ that is relatively prime to the exponent of Z. Then there is a canonical morphism of functors $\bar{r}_{\widetilde{G}}$ such that the diagram*

$$
\begin{array}{ccc}
H^1(*, \widetilde{G}/Z) & \xrightarrow{\ \bar{r}_{\widetilde{G}}\ } & H^3(*, \mathbb{Z}/m\mathbb{Z}(2)) \\
\uparrow & & \uparrow{\scriptstyle \pi} \\
H^1(*, \widetilde{G}) & \xrightarrow{\ r_{\widetilde{G}}\ } & H^3(*, \mathbb{Z}/n_{\widetilde{G}}\mathbb{Z}(2))
\end{array}
$$

commutes, where π is the projection arising from the Chinese remainder decomposition of $H^3(*, \mathbb{Z}/n_{\widetilde{G}}\mathbb{Z}(2))$.

Under the additional hypothesis that m is not divisible by $\mathrm{char}\, k$, this result was proved in [18, Prop. 7.2] using the elementary theory of cohomological invariants from [15]. We give a proof valid for all characteristics that relies on the (deeper) theory of invariants of degree 3 of semisimple groups developed in [29].

Proof. Put $G := \widetilde{G}/Z$. For G of inner type, this result is included in the calculations in [29, §4]. Consulting the list of Dynkin indexes for groups of outer type from [20], we have $m = 1$ except for types ${}^2A_n$ with n even, types ${}^3D_4$ or ${}^6D_4$, and type ${}^2E_6$, where m is respectively 2, 3 and 4. To complete the proof, we calculate the group denoted $Q(G)/\mathrm{Dec}(G)$ in [29]. Note that $Q(G)$ may be calculated over an algebraic closure, so the calculations in [29] show that $Q(G)$ is $(n+1)\mathbb{Z}q$, $2\mathbb{Z}q$, or $3\mathbb{Z}q$ respectively.

The group $\mathrm{Dec}(G)$ is $n_G \mathbb{Z}q$ where n_G is the gcd of the Dynkin index of each representation ρ as ρ varies over the k-defined representations of G. Clearly, n_G is unchanged by replacing G by a twist by a 1-cocycle $\eta \in H^1(k, \widetilde{G})$, and $n_{\widetilde{G}_\eta}$ divides n_{G_η}. Consulting then the maximum values for $n_{\widetilde{G}_\eta}$ from [20], we conclude that n_G is divisible by $2(n+1)$, 12, 12 respectively. On the other hand, n_G divides the Dynkin index of the adjoint representation, which is twice the dual Coxeter number; hence n_G divides $2(n+1)$, 12, 24 respectively. Thus the maximal divisor of $|Q(G)/\mathrm{Dec}(G)|$ that is relatively prime to the exponent of Z divides m and the claim follows from the main theorem of [29].

For completeness, we note that for type $^2E_6$, the Weyl module with highest weight $\lambda_1 + \lambda_6$ of dimension 650 has Dynkin index 300 [27], so n_G divides $\gcd(24, 300) = 12$, i.e., $n_G = 12$. $\square$

DEFINITION 4. Suppose G is an absolutely almost simple algebraic group, and put $\widetilde{G}$, $\bar{G}$ for its simply connected cover and adjoint quotient. For m as defined in Lemma 3, we define:

$$b(G) := -\bar{r}_{\widetilde{G}}(\nu_{\bar{G}}) \quad \in H^3(k, \mathbb{Z}/m\mathbb{Z}(2)).$$

If $t_G = 0$, we define:

$$a(G) := -r_{\widetilde{G}}(\xi_{\bar{G}}) \quad \in H^3(k, \mathbb{Z}/n_{\widetilde{G}}\mathbb{Z}(2))$$

for $\xi_{\bar{G}}$ as in Lemma 1.

Factoring $n_{\widetilde{G}} = mc$, we have $H^3(k, \mathbb{Z}/n_{\widetilde{G}}\mathbb{Z}(2)) = H^3(k, \mathbb{Z}/m\mathbb{Z}(2)) \oplus H^3(k, \mathbb{Z}/c\mathbb{Z}(2))$. In case $t_G = 0$, by Lemma 3 the invariants are related by the equation $a(G) = b(G) + c(G)$ for some $c(G) \in H^3(k, \mathbb{Z}/c\mathbb{Z}(2))$.
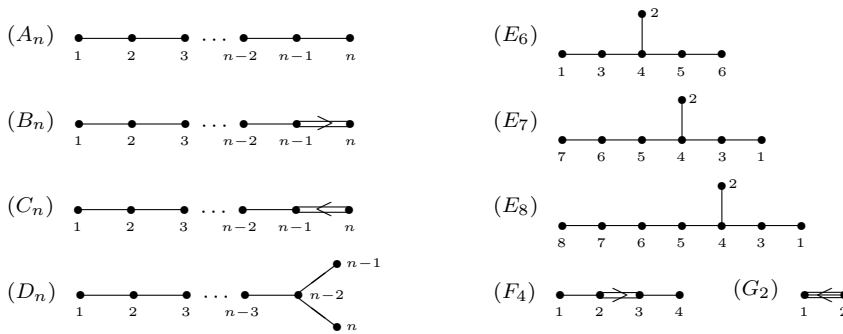

TABLE 2. Dynkin diagrams of simple root systems, with simple roots numbered

2. Tits p-indexes of classical groups

As envisioned by Siegel and Weil [50] and detailed in [25], classical groups can be described over a field k of characteristic $\neq 2$ as automorphism groups of central simple algebras with involutions. Recall that a simple k-algebra is said to be central if its k-dimension is finite and if its center is k. The degree $\deg(A)$ of a central simple algebra is the square root of its dimension and its index $\mathrm{ind}(A)$ is the degree of a division k-algebra Brauer equivalent to A.

An involution σ on a central simple algebra A is a k-linear antiautomorphism of order 2. Following [25], the index $\mathrm{ind}(A, \sigma)$ of a central simple algebra is the set of the reduced dimensions of the σ-isotropic right ideals in A. (Recall that a right ideal I of A is σ-isotropic if $\sigma(I)I = 0$.) As [25, §6] states, $\mathrm{ind}(A, \sigma)$ is of the form $\{0, \mathrm{ind}(A), ..., r\,\mathrm{ind}(A)\}$, where r is the Witt index of an (skew-)hermitian form attached to (A, σ).

DEFINITION 5.   Let p be a prime and (A, σ) be a central simple k-algebra with involution. The *p-index* of (A, σ) is the union of the sets $\mathrm{ind}(A \otimes_k E, \sigma \otimes \mathrm{Id_E})$, where E runs through all finite extensions of k of degree not divisible by p; we denote it by $\mathrm{ind}_p(A, \sigma)$.

Entirely analogous statements hold for a central simple algebra with quadratic pair (A, σ, f) as defined in [**25**]; one simply adapts the notion of isotropic right ideal and $\mathrm{ind}(A, \sigma, f)$ as in [**25**, pp. 73, 74].

In this section we will list the possible Tits p-indexes for the classical groups and relate them to the index of a corresponding central simple algebra with involution or quadratic pair. The actual list of indexes is identical with the one in [**47**]; the core new material here is the proof of existence of such indexes over 2-special fields given in the next section. We do not discuss motivic equivalence for classical groups here, because the cases of special orthogonal groups and groups of type ^1A_n were handled in [**8**].

2.1. Type ^1A_n.

An absolutely simple simply connected groups of type ^1A_n is isomorphic to the special linear group $\mathrm{SL}_1(A)$ of a degree $n + 1$ central simple k-algebra A. The coprime-to-p components of A vanishes over a p-special closure of k, hence denoting by d_p the integer $p^{v_p(\mathrm{ind}(A))}$ we get the following description of the p-indexes of type ^1A_n. In the interest of clarity, we list after the Tits p-index the distinguished orbits in the p-index, which we call *distinguished p-orbits* to emphasize the prime p.


$(^1A_n, p)$
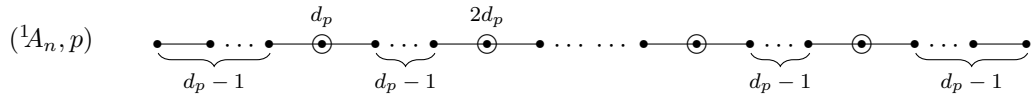
Distinguished p-orbits: $\{d_p, 2d_p, ..., n + 1 - d_p\}$.

The twisted flag varieties of type ^1A_n are isomorphic to the varieties of flags of right ideals of fixed dimension [**30**]. Note that any power of p may be realized as the integer d_p for a central simple algebra defined over a suitable field k.

2.2. Type ^2A_n.

The absolutely simple simply connected groups of type ^2A_n correspond to the special unitary groups $\mathrm{SU}(A, \sigma)$, where (A, σ) is a central simple algebra of degree $n + 1$ with involution of the second kind. (Recall that in this case A is a central simple K-algebra for a separable quadratic extension K of k.) As in 1.4, we need only consider the prime 2. We denote by d the index of A and r is the integer such that $\mathrm{ind}_2(A, \sigma) = \{d_2, 2d_2, ..., rd_2\}$.


$(^2A_n, 2)$
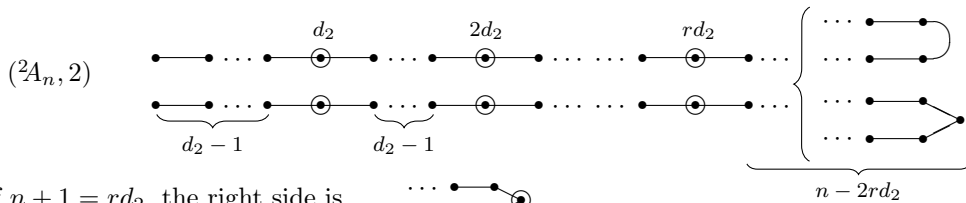
if $n + 1 = rd_2$, the right side is

Distinguished 2-orbits: $\{\{d_2, n + 1 - d_2\}, \{2d_2, n + 1 - 2d_2\}, ..., \{rd_2, n + 1 - rd_2\}\}$.

The associated twisted flag varieties are described in [**30**]. We will show in §3 that any of such Tits 2-index can be realized by a group of type ^2A_n defined over a suitable field.

2.3. Type B_n.

An absolutely simple simply connected group of type B_n is isomorphic to the spinor group $\mathrm{Spin}(V,q)$ of a $(2n+1)$-dimensional quadratic space (V,q). (The adjoint groups of type B_n correspond to the special orthogonal groups.) The *Witt index* of the quadratic space (V,q) is denoted by $i_w(q)$ and the only torsion prime here is 2.


$(B_n, 2)$
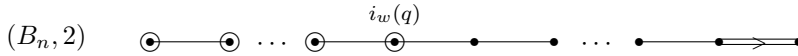
Distinguished 2-orbits: $\{1, 2, ..., i_w(q)\}$.

The Tits 2-index coincides with the classical Tits index for such groups by Springer's theorem (see [**44**], [**9**, Corollary 18.5] for a characteristic-free proof). It is thus known from Tits classification that any such Tits 2-index can be achieved as the index of the spinor group of some quadratic space (V,q).

2.4. Type C_n.

Up to isomorphism, an absolutely simple simply connected group of type C_n is the symplectic group $\mathrm{Sp}(A,\sigma)$ associated to a central simple k-algebra of degree $2n$ with symplectic involution. As previously the only torsion prime here is 2 and we denote the 2-index $\mathrm{ind}_2(A,\sigma) = \{d, 2d, ..., rd\}$, where d is the index of A.


$(C_n, 2)$
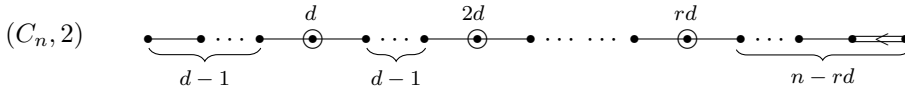
if $n = rd$, the right side becomes

Distinguished 2-orbits: $\{d, 2d, ..., rd\}$.

The twisted flag varieties for such groups correspond to varieties of flags of σ-isotropic subspaces of fixed dimension [**30**]. We will show in §3 how to construct symplectic groups of any such prescribed 2-index over suitable fields.

2.5. Type ^1D_n.

Over a base field k of characteristic $\neq 2$, absolutely simple simply connected algebraic group of type ^1D_n are described as spinor groups $\mathrm{Spin}(A,\sigma)$ of a $2n$-degree central simple k-algebras (A,σ) with orthogonal involution of trivial discriminant. The suitable generalization to include k of characteristic 2 is the notion of algebra with quadratic pair (A,σ,f) as in [**25**]. We keep the same notations as before for the indexes of A, (A,σ), and (A,σ,f).

$(^1D_n, 2)$

$d = 1$ and $n = r$

$d = 2^k$, $k \geq 2$ and $n = rd$

$d = 2$ and $n = 2r$

$n - rd = 2$
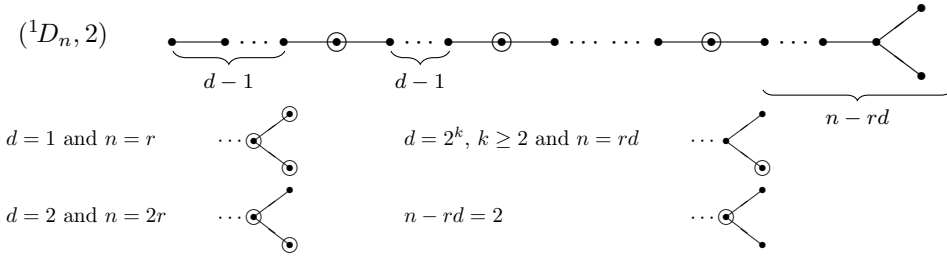
Distinguished 2-orbits: $\{d, 2d, ..., rd\}$.

As detailed in [**30**] when char $k \neq 2$, the associated twisted flag varieties are the varieties of flags of σ-isotropic subspaces of prescribed dimension. This same description holds in all characteristics (replacing (A, σ) with (A, σ, f)), as can be seen in case A is split as in [**2**, pp. 258–262] and by Galois descent for general A, cf. [**6**, p. 219].

Any such 2-index can be realized as the index of the spinor group of a central simple k-algebra with orthogonal involution of trivial discriminant over a suitable field.

2.6. Type ^2D_n.

Absolutely simple simply connected algebraic groups of type ^2D_n are described by spinor groups of $2n$-degree central simple k-algebras endowed with an orthogonal involution of non-trivial discriminant (again, this notion is replaced by quadratic pairs to cover base fields of characteristic 2). We denote here the discrete invariants associated to algebras with involution in the same way as for ^1D_n, and the only torsion prime is 2. As for the ^1D_n case, the twisted flag varieties for such groups are described in [**30**] and for any such prescribed 2-index can be associated to a suitable spinor group.

$(^2D_n, 2)$

$n - rd = 1$ and $d = 1$

$n - rd = 1$ and $d = 2$

$n - 2rd$
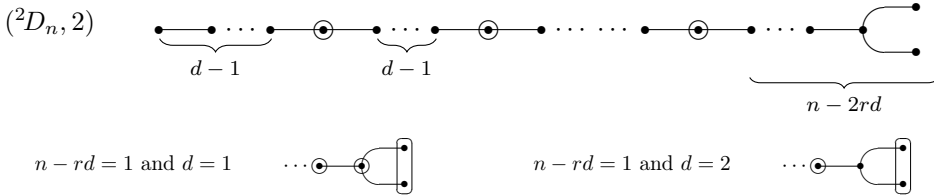
Distinguished 2-orbits: $\begin{cases} \{d, 2d, ..., rd\} & \text{if } rd < n - 1; \\ \{d, 2d, ..., (r-1)d, \{n-1, n\}\} & \text{if } rd = n - 1. \end{cases}$

3. Tignol's construction

To complete the determination of all the values of the Tits p-indexes of classical groups, it remains to show that each of the previously-announced indices can be realized by suitable absolutely simple groups. Recall that a central simple algebra with involution (A, σ) is adjoint to a (skew)-hermitian form h_σ on a right D-module, where D is a division algebra Brauer-equivalent to A. Adding hyperbolic planes to h_σ, the problem is reduced to the construction of *anisotropic* central simple algebras with involutions (A, σ) of any kind with A of any index a power of 2 over 2-special fields. We reproduce here a construction of such algebras with involutions which is due to Jean-Pierre Tignol.

Let Γ_n be a product of n copies of $\mathbb{Z}_{(2)}$, the ring of rational numbers with odd denominators. For any field K, consider the field K_n of power series $\sum_{\gamma \in \Gamma_n} a_\gamma x^\gamma$ whose support is well ordered

with respect to the lexicographical order [**10**]. The field K_n is endowed with the valuation $v : K_n \longrightarrow \Gamma_n \cup \{\infty\}$ which sends an element to the least element of its support.

LEMMA 6.   *If K is 2-special, then K_n is also 2-special.*

Proof.   Let L be a finite separable field extension of K_n. The valuation v extends uniquely to L and K_n is maximally complete, hence the equality

$$[L : K_n] = [\bar{L} : K] \cdot (v(L^\times) : v(K_n^\times))$$

holds [**42**, Ch. 2]. The field K being assumed to be 2-special, $[\bar{L} : K]$ is a power of 2. Moreover the quotient group $v(L^\times)/v(K_n^\times)$ is torsion [**10**, Theorem 3.2.4] and $v(K_n^\times)$ is Γ_n, hence the order of the quotient group $v(L^\times)/v(K_n^\times)$ is a power of 2. $\square$

We now describe Tignol's procedure to construct from any K-division algebra with involution (D, σ) a family of anisotropic algebras with involutions of the same kind over K_n.

PROPOSITION 7.   *Let M be a right D_{K_n}-module of rank k which is at most n. The hermitian form*

$$h_k \quad : \qquad M \times M \qquad \longrightarrow \qquad D_{K_n}$$
$$(a_1, ..., a_k, b_1, ...b_k) \quad \mapsto \quad \sum_{i=1}^{k} \sigma_{K_n}(a_i)x^{\varepsilon_i}b_i$$

where ε_i is the n-uple whose only non-zero entry is 1 at the i-th position is anisotropic.

Proof.   Setting $v(d \otimes \lambda) = v(\lambda)$ for any $d \in D^\times$ and $\lambda \in K_n^\times$, the valuation v extends to a σ-invariant valuation on D_{K_n}. One observes that for any element a of $D_{K_n}^\times$, $v(\sigma_{K_n}(a)x^{\varepsilon_i}a) = \varepsilon_i + 2v(a)$ belongs to $\varepsilon_i + 2\Gamma_n$ and thus

$$v\left(\sum_{i=1}^{k} \sigma_{K_n}(a_i)x^{\varepsilon_i}a_i\right) = \min\{\varepsilon_i + 2v(a_i), i = 1, ..., k\}.$$

It follows that if $h_k(a_1, ..., a_k, a_1, ..., a_k) = 0$, then $a_i = 0$ for all $i = 1, ..., k$. $\square$

COROLLARY 8.   *Each of the previously described indices of type $(^2A_n, 2)$, $(C_n, 2)$, $(^1D_n, 2)$, $(^2D_n, 2)$ is the Tits 2-index of a semisimple algebraic group defined on a suitable field.*

Proof.   As seen in a previous discussion, it suffices to construct over 2-special fields algebras with anisotropic involutions (A, σ) of any kind, where the index of A can be any power of 2.

Take a sufficiently large transcendental field extension $K(x_1, ..., x_s)$ of a field K, over which we can consider a division algebra with involution (D, σ) of any kind. (For instance, D may be chosen to be a tensor product of quaternion algebras.) Writing L for the 2-special closure of $K(x_1, ..., x_s)$, we can apply Tignol's procedure to (D_L, σ_L). Proposition 7 gives rise to an anisotropic algebra with involution $(M_k(D_{L_n}), \sigma_{L_n})$ which is of the same kind as (D, σ) and thus fulfills the required assumptions. $\square$

4.   *Tits p-indexes of exceptional groups*

In this section we will list the possible Tits p-indexes of *exceptional* groups, meaning groups of the types omitted from §2.

4.1.   *The cases $(G_2, 2)$, $(^3D_4, 3)$, $(F_4, 3)$, and $(E_8, 5)$.*

We now consider some groups G relative to a prime p in cases where p does not divide the exponent of the center of the simply connected cover $\widetilde{G}$ and the Dynkin index $n_{\widetilde{G}}$ factors as cp for some c not divisible by p. Definition 4 then gives an element $b(G) \in H^3(k, \mathbb{Z}/p\mathbb{Z}(2))$ depending only on G.

PROPOSITION 9.   *If the quasi-split type of G and p are one of $(G_2, 2)$, $(^3D_4, 3)$, $(F_4, 3)$, or $(E_8, 5)$, then the following are equivalent:*
  (i) *G is quasi-split by a finite separable extension of k of degree not divisible by p.*
  (ii) *G is isotropic over a finite separable extension of k of degree not divisible by p.*
  (iii) *$b(G) = 0$.*
And for G of type G_2, F_4, or E_8 the preceding are equivalent also to:
  (iv) *The Chow motive with $\mathbb{F}_p$ coefficients of the variety of Borel subgroups of G is a sum of Tate motives.*
Moreover, for every field k, there exists a p-special field $F \supseteq k$ and an anisotropic F-group of the same quasi-split type as G.

Proof.   It is harmless to assume that k is p-special. Suppose (ii), that G is k-isotropic. When we consult the list of possible Tits indexes from [**47**], we find that for G of type G_2, G is necessarily split. If G has type $^3D_4$, then the only other possibility is that the semisimple anisotropic kernel is isogenous to a transfer $R_{L/k}(A_1)$ where L is cubic Galois over k. But L has no separable field extensions of degree 2, so a group of type A_1 is isotropic, hence G is split. If G has type F_4, the only possibility has type B_3, which is isotropic, hence again (i). If G has type E_8, the possibilities are E_7, D_7, E_6, D_6, or D_4, and all such groups are isotropic as in Table 1. Thus, (ii) implies (i).

Assume now (iii); we prove (i). As k is p-special (and because of our choice of G), $t_G = 0$, so we are reduced to showing that, for $\widetilde{G}^q$ the quasi-split inner form of the simply connected cover of G, the Rost invariant $r_{\widetilde{G}^q}$ has zero kernel. By the main result of [**24**], we may assume that char $k = 0$. The kernel is zero for type G_2 by [**20**, p. 44], for F_4 it is [**25**, §40], for $^3D_4$ it is [**16**], for E_8 it is [**7**] or see [**15**, 15.5].

Trivially, (i) implies the other conditions, including (iv). The remaining implication, that (iv) implies (i), is [**37**, Cor. 6.7].

For existence, choosing a versal torsor under the simply connected cover of G provides an extension $E \supseteq k$ and an E-group G' of the same quasi-split type as G with $b(G') \neq 0$. Then $G' \times F$ is the desired group, where F is any p-special closure of E. $\square$

For (G, p) as in the proposition, we needn't display the possible Tits p-indexes, because there are only two possibilities: quasi-split (every vertex is distinguished) or anisotropic (no vertex is distinguished).

COROLLARY 10.   *Suppose (G, p) is one of the pairs considered in Proposition 9, and G' is a simple algebraic group that is an inner form of G. Then G and G' are motivic equivalent mod p if and only if $b(G)$ and $b(G')$ generate the same subgroup of $H^3(k, \mathbb{Z}/p\mathbb{Z}(2))$.*

Proof.   The element $\nu_G \in H^1(k, G)$ represents a principal homogeneous space; write K for its function field. The kernel of $H^3(k, \mathbb{Z}/p\mathbb{Z}(2)) \to H^3(K, \mathbb{Z}/p\mathbb{Z}(2))$ is the group generated by $b(G)$ [**20**, p. 129, Th. 9.10]. If the isomorphism of the Tits p-indexes extends to an isomorphism between p-indexes of G_E and G'_E for every extension E of k, then G_K and G'_K are both

quasi-split, hence $b(G')$ is in $\langle b(G) \rangle$; by symmetry the two subgroups generated by $b(G)$ and $b(G')$ are equal. Conversely, if the two subgroups are equal, then for each extension E of k, either $\mathrm{res}_{E/k}(b(G))$ is zero and both G_E and G'_E are quasi-split, or it is nonzero and both are anisotropic; in this case the isomorphism of the Tits p-indexes over k clearly extends to an isomorphism over E. Applying the main result of [8] gives the claim. $\qquad\square$

Remarks specific to G_2. For G, G' of type G_2, we have: $G \cong G'$ iff $b(G) = b(G')$, cf. [25, 33.19] and [46], i.e., motivic equivalence mod 2 is the same as isomorphism.

The flag varieties for type G_2 are described in [6, Example 9.2].

Remarks specific to F_4 at $p = 3$. For G, G' of type F_4 over a 3-special field k, we have: $G \cong G'$ iff $b(G) = b(G')$. If char $k = 0$, this is [41] and we can transfer this result to all characteristics using the same method as in [22, §9].

4.2. Type D_4.

For G of type D_4, $\mathrm{Aut}(\Delta)(k_{\mathrm{sep}})$ is the symmetric group on 3 letters, so G has type ${}^t D_4$ with $t = 1, 2, 3,$ or 6. Groups of type ${}^1 D_4$ or ${}^2 D_4$ were treated in §2; this includes the case where k is 2-special. Thus it remains to consider groups of type ${}^3 D_4$ and $p = 3$, which was treated in subsection 4.1.

Flag varieties. For groups of type ${}^3 D_4$ and ${}^6 D_4$, the k-points of the twisted flag varieties are described in [12].

4.3. Type F_4 and $p = 2$.

Groups of type F_4, just as for type G_2, are all simply connected and adjoint and therefore all have Tits class zero; therefore the invariants a and b from Definition 4 agree. For G a group of type F_4, one traditionally decomposes $b(G) \in H^3(k, \mathbb{Z}/6\mathbb{Z}(2))$ as $f_3(G) + g_3(G)$ for $f_3(G) \in H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$ and $g_3(G) \in H^3(k, \mathbb{Z}/3\mathbb{Z}(2))$. There is furthermore another cohomological invariant $f_5(G) \in H^5(k, \mathbb{Z}/2\mathbb{Z}(4))$, see [25, 37.16] or [20, p. 50] when char $k \neq 2$ (in which case $f_5(G)$ belongs to $H^5(k, \mathbb{Z}/2\mathbb{Z})$) or [34, §4] for arbitrary k. (These statements rely on viewing each group of type F_4 over k as the automorphism group of a uniquely determined Albert k-algebra. For general background information on Albert algebras, see [25, Ch. IX], [46], or [35].) Table 3 gives a dictionary relating the Tits index of G with the values of these invariants; in the last column we give the signature of the Killing form for the Lie algebra over $\mathbb{R}$ with that Tits index. (Implicitly this is a statement of existence; one can calculate the signature of the Killing form from the Tits index by the formula from [26, §6].)

Tits index of G	$f_3(G)$	$f_5(G)$	$g_3(G)$	signature of real form
⊙—⊙⟩⊙—⊙	0	0	0	4
•—•⟩•—⊙	$\neq 0$	0	0	-20
•—•⟩•—•	$f_5(G)$ and $g_3(G)$ not both zero			-52

TABLE 3. *Tits index of a group of type F_4*

For type F_4, we should consider $p = 2, 3$ by Table 1. The case $p = 3$ was handled in Proposition 9. For $p = 2$, all three possible indexes occur over the 2-special field $\mathbb{R}$, so they are also 2-indexes. Alternatively, one can handle the $p = 2$ case by noting that groups of type F_4 over a 2-special field are of the form $\mathrm{Aut}(J)$ for an Albert algebra J containing a nonzero element with norm zero, i.e., such that J is reduced as described in [34, 1.7] or [20, p. 47].

PROPOSITION 11.   *Groups G and G' be groups of type F_4 over a field k are motivic equivalent mod 2 iff $f_3(G) = f_3(G')$ and $f_5(G) = f_5(G')$. The groups G, G' are motivic equivalent (mod every prime) iff $f_3(G) = f_3(G')$, $f_5(G) = f_5(G')$, and $g_3(G) = \pm g_3(G')$.*

Proof.   The elements $f_d(G)$ for $d = 3, 5$ are symbols, so we may find d-Pfister quadratic forms q_d whose Milnor invariant $e_d(q_d) \in H^d(k, \mathbb{Z}/2\mathbb{Z}(d-1))$ equals $f_d(G)$. For K_d the function field of q_d, the kernel of the map $H^d(k, \mathbb{Z}/2\mathbb{Z}(d-1)) \to H^d(K_d, \mathbb{Z}/2\mathbb{Z}(d-1))$ is generated by $f_d(G)$ as follows from [**33**, Th. 2.1] (if char $k \neq 2$) as explained in [**9**, p. 180]. The first claim now follows by the arguments used to prove Corollary 10. The second claim follows from the first and Corollary 10. $\square$

We thank Holger Petersson for contributing the following example.

EXAMPLE 12.   Given a group G of type F_4 over k, we describe how to construct a group G' that is motivic equivalent to G modulo every prime but which need not be isomorphic to G.

Write $G = \mathrm{Aut}(J)$ and furthermore write J as a second Tits construction Albert algebra $J = J(B, \tau, u, \mu)$ for some quadratic étale k-algebra K, where B is a central simple K-algebra of degree 3 with unitary K/k-involution τ, $u \in B^\times$ is such that $\tau(u) = u$, and $\mu \in K^\times$ satisfies $N_{K/k}(\mu) = \mathrm{Nrd}_B(u)$. Then $g_3(G) = -\mathrm{cor}_{K/k}([B] \cup [\mu])$ and $f_3(G)$ is the 3-Pfister quadratic form over k corresponding to the unitary involution $\tau^{(u)} : x \mapsto u^{-1}\tau(x)u$ on B. The 1-Pfister $N_{K/k}$ is a subform of the 3-Pfister q corresponding to the involution τ [**36**, Prop. 2.3] so there exist $\gamma_1, \gamma_2 \in k^\times$ such that $q = \langle\!\langle \gamma_1, \gamma_2 \rangle\!\rangle \otimes N_{K/k}$. Combining [**36**, 2.9] and [**35**, 7.9] gives

$$f_5(G) = \langle\!\langle \gamma_1, \gamma_2 \rangle\!\rangle \otimes f_3(G). \tag{4.1}$$

Define now $G' := \mathrm{Aut}(J')$ where J' is the second Tits construction Albert algebra $J(B, \tau, u^{-1}, \mu^{-1})$. Since the unitary involutions $\tau^{(u)}$ and $\tau^{(u^{-1})}$ of B are isomorphic under the inner automorphism $x \mapsto uxu^{-1}$, we have $f_3(G') = f_3(G)$. As $\langle\!\langle \gamma_1, \gamma_2 \rangle\!\rangle$ does not change when passing from J to J', we find $f_5(G') = f_5(G)$. As clearly $g_3(G') = -g_3(G)$, G and G' are motivic equivalent mod p for all p.

It is unknown if J' depends on the choice of expression of J as a second Tits construction; perhaps it only depends on J. This is a specific illustration of the general open problem [**43**, p. 465]: Do the invariants f_3, f_5, and g_3 distinguish groups of type F_4?

Flag varieties.   For groups of type F_4, the k-points of the twisted flag varieties are described in [**6**, 9.1], relying on [**40**] or [**1**]. A portion of this description for $k = \mathbb{R}$ can be found in [**11**, 28.22, 28.27]. For J an Albert algebra, $\mathrm{Aut}(J)$ is isotropic iff J has nonzero nilpotents, and $\mathrm{Aut}(J)$ is split iff J is the split Albert algebra.

4.4.   *Type E_6 and $p = 2, 3$.*

For G of type $^1E_6$, the class t_G has order dividing 3 and can be represented by a central simple algebra of degree 27 [**48**, p. 213] which we denote by A; it is only defined up to interchanging with its opposite algebra. The list of possible Tits indexes from [**47**] is reproduced in the first column of Table 4. The constraints on the index of A given in the second column can be deduced from the possible indexes of the Tits algebras of the semisimple anisotropic kernel as explained in [**48**, p. 211]. In the column for 2-special fields, we give the signature of the Killing form on the real Lie algebra if one occurs with that Tits index.

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index of G	ind A	occurs as a 2-index?	occurs as a 3-index?
	1	yes (6)	yes
	1	yes (-26)	no
	3	no	yes
	divides 27	no	yes

TABLE 4. *Possible Tits indexes of groups of type* $^1E_6$

4.4.1. *Type* $^1E_6$ *and* $p = 3$. Over a 3-special field, every group of type $^1D_4$ is split, therefore the Tits index with semisimple anisotropic kernel of that type (row 2 in Table 4) cannot occur. Table 5 is justified in [**22**, §10], where the top row — which is only proved assuming char $k = 0$ — refers to the mod-3 J-invariant defined in [**37**] describing the decomposition of the mod-3 Chow motive of the variety X_Δ of Borel subgroups of G. In particular, there exists an anisotropic group of type $^1E_6$ over a 3-special field, completing the justification of the last column of Table 4.

$J_3(G)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	$(2,1)$
Tits 3-index of G	split		$\cdots$ anisotropic $\cdots$		
index of A	1	3	1	3	9 or 27

TABLE 5. *Possible values for the mod-3 J-invariant of G of type* $^1E_6$, *assuming* char $k = 0$

4.4.2. *Type* $^1E_6$ *with* $t_G = 0$. For any group G of type $^1E_6$ with $t_G = 0$, we get from Definition 4 an element $a(G) \in H^3(k, \mathbb{Z}/6\mathbb{Z}(2))$, which we write as $f_3(G) + g_3(G)$ for $f_3(G) \in H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$ and $g_3(G) \in H^3(k, \mathbb{Z}/3\mathbb{Z}(2))$. It follows from [**15**, 11.1] that the simply connected cover of G is the group of isometries of the cubic norm form of an Albert algebra J, and from general properties of the Rost invariant that $f_3(G)$ and $g_3(G)$ equal the corresponding values for the automorphism group $\mathrm{Aut}(J)$ of type F_4. Combining the description of the flag varieties of G in terms of subspaces of J from [**6**, §7] as well as the relationships between values of the cohomological invariants and properties of J from [**25**, §40] and [**34**] gives the information in Table 6.

PROPOSITION 13. *Let G and G' be groups of type $^1E_6$ over a field k such that $t_G = t_{G'} = 0$. Then G and G' are motivic equivalent modulo a prime p if and only if the p-torsion components of $a(G)$ and $a(G')$ generate the same subgroup of $H^3(k, \mathbb{Z}/p\mathbb{Z}(2))$.*

Proof. Table 6 shows that the Tits 2-index only depends on whether $f_3(G)$ is zero and the 3-index only depends on whether $g_3(G)$ is zero. Combine this with the main result of [**8**]. $\square$
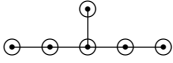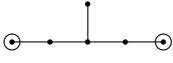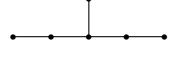
Tits index of G	$f_3(G)$	$g_3(G)$	occurs as a 2-index?	occurs as a 3-index?
	0	0	yes	yes
	$\neq 0$	0	yes	no
	any	$\neq 0$	no	yes

TABLE 6. *Table of possible Tits indexes for G of type $^1E_6$ with $t_G = 0$*

4.4.3. *Type $^1E_6$ and $p = 2$.* Over a 2-special field, the Tits class of any group G of type E_6 is zero so Table 6 applies and in particular G is isotropic. From this, we deduce the third column of Table 4.

COROLLARY 14. *Groups G and G' of type $^1E_6$ over a field k are motivic equivalent mod 2 if and only if $f_3(G) = f_3(G')$.*

Proof. As the Tits classes t_G and $t_{G'}$ are zero over every 2-special field, the claim follows immediately from Proposition 13. □

4.4.4. *Type $^2E_6$ and $p = 2$.* For G of type $^2E_6$, the element $b(G)$ belongs to $H^3(k, \mathbb{Z}/4\mathbb{Z}(2))$. (If k is 2-special, then $t_G = 0$ and $a(G) = b(G)$.) In this setting, the possible Tits 2-indexes have been determined in [**21**, Prop. 2.3]. We reproduce that table here, as well as indicate the signature of the Killing form on the real simple Lie algebra with that Tits index, if such occurs.
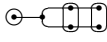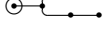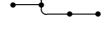
index	$b(G) \in H^3(k, \mathbb{Z}/4\mathbb{Z}(2))$	occurs over $\mathbb{R}$?
	0	yes (2)
	nonzero symbol in $H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$ killed by K	yes (-14)
	symbol in $H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$ not killed by K	no
	in $H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$, not a symbol	no
	$\neq 0$	yes (-78)

TABLE 7. *Possible Tits 2-indexes for G of type $^2E_6$. The field K is the separable quadratic extension of k such that $G \times K$ has type $^1E_6$.*

Flag varieties. The k-points of the twisted flag varieties for groups of type $^1E_6$ are described in [**6**, §7] and for type $^2E_6$ in [**21**, §5]. If G is a group of type $^1E_6$ with $t_G = 0$, then the simply connected cover of G is the the group of norm isometries of an Albert k-algebra J and the variety of total flags has k-points $\{S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5\}$ where $S_i \subset J$ is a singular subspace, $\dim S_i = i$, and S_5 is not a maximal singular subspace.

4.5. *Type E_7 and $p = 2, 3$.*

Let now G have type E_7. The class t_G has order 1 or 2 and can be represented by a unique central simple algebra of degree 8 [**48**, 6.5.1], which we denote by A. Table 8 lists the possible Tits indexes for G. As before, if a Tits index occurs over $\mathbb{R}$, we list the signature of the Killing form in the 2-index column.

index of G	ind A	occurs as a 2-index?	occurs as a 3-index?
	1	yes (7)	yes
	2	yes (−5)	no
	1	yes (−25)	no
	2	yes	no
	2	yes	no
	divides 4	yes	no
	1	no	yes
	divides 8	yes (−133)	no

TABLE 8. *Tits indexes of groups of type E_7*

4.5.1. *The index of A*   The second column of Table 8 lists the possible values for the index of A, taken from [**48**, 6.5.5]. In that reference, Tits asked whether the groups in the sixth row of Table 8, those with semisimple anisotropic kernel of type $^1D_6$, could be explicitly described. We now do so under the assumption that char $k \neq 2$ and using the language of [**25**]. By Tits's Witt-type Theorem, it is equivalent to describe the semisimple anisotropic kernel up to isogeny, meaning a group $\mathrm{SO}(A, \sigma)$ where A is a central simple k-algebra of degree 12 and exponent 2 — in particular $A \cong M_3(D)$ for some D of degree 4 — σ is an orthogonal involution with trivial discriminant, and the even Clifford algebra $C(A, \sigma)$ is $M_{32}(k) \times M_8(D)$. All such (A, σ) are obtained by the construction in the paper [**23**], which takes as inputs a quadratic étale k-algebra K, a central simple K-algebra B of degree 6 and exponent 2, and a unitary involution τ on B. That is, every such (B, τ) produces an (A, σ) and thereby a group of type E_7 with semisimple anisotropic kernel of type $^1D_6$ or with more circled vertices, and every such E_7 is obtained in this way. To explicitly give an E_7 with ind $A = 4$, it suffices to pick a (B, τ) whose discriminant algebra has index 4, which is done for example in [**25**, p. 148, Exercise 13].

4.5.2. *Type E_7 and $p = 2$.*   We must justify the third column of Table 8. As in §4.4.3, a group of type $^1E_6$ over a 2-special field is isotropic, so a group of type E_7 over a 2-special field cannot have semisimple anisotropic kernel of type $^1E_6$.

For the three "yes" entries that do not occur over $\mathbb{R}$, we note that group of type E_7 with these Tits indexes occur over some field k and odd-degree extensions of k cannot make the semisimple anisotropic kernel (of type ${}^1D_4 \times {}^1A_1$, ${}^1D_5 \times {}^1A_1$, or ${}^1D_6$) isotropic.

In the anisotropic case in the last row of the table, A may have index 1, 2, 4, or 8 over a 2-special field. A versal form of E_7 over some field F has Tits algebra of index 8 by [**31**, p. 164], [**28**], or [**18**, Lemma 14.3]. Going up to a 2-special extension of F, it is clear that there do exist groups of type E_7 over a 2-special field that have Tits algebra of index 8 and so are anisotropic. The compact real form of E_7 has Tits algebra the quaternions (of index 2). An example of an anisotropic group G of type E_7 with $t_G = 0$ over a 2-special field is given in [**15**, Example A.2].

4.5.3. *Type E_7 and $p = 3$.* We now justify Table 9, which implies the claims in the fourth column of Table 8. Let G be a group of type E_7 over a 3-special field k, so $t_G = 0$.

We claim that G is isotropic. Suppose first the char $k \neq 2, 3$. Put E_6 and E_7 for the split simply connected groups of those types. The natural inclusion $E_6 \subset E_7$ gives a surjection in cohomology $H^1(k, E_6 \rtimes \mu_4) \to H^1(k, E_7)$, see [**15**, 12.13]. As $H^1(k, \mu_4) = 0$, the class ξ_G from Lemma 1 lies in the image of $H^1(k, E_6)$ and it follows that the Tits index of $(E_7)_{\xi_G}$, i.e., of the simply connected cover of G, is as in the bottom row of Table 9 or has more distinguished vertices. Note that as in Table 6, $(E_6)_{\xi_G}$ is split iff $g_3((E_6)_{\xi_G}) = 0$ iff $b(G) = 0$, and anisotropic iff $g_3((E_6)_{\xi_G}) \neq 0$ iff $b(G) \neq 0$. This completes the justification of Table 9 if char $k \neq 2, 3$. If k has characteristic 2 or 3, then arguing as in [**22**, §9] reduces the claim to the case of characteristic zero.
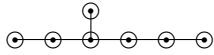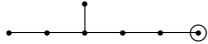

TABLE 9. *Possible Tits 3-indexes for a group G of type E_7*

PROPOSITION 15. *Simple algebraic groups G and G' of type E_7 over a field k are motivic equivalent mod 3 iff $b(G) = \pm b(G') \in H^3(k, \mathbb{Z}/3\mathbb{Z}(2))$.*

Proof. Combine Table 9 and the arguments used for Corollary 10.   □

Flag varieties. The flag varieties for groups of type E_7 are described in [**13**, §4] and [**14**].

4.6. *Type E_8 and $p = 2, 3$.*

For type E_8, by Table 1 we should consider $p = 2, 3, 5$. The case $p = 5$ was handled in Proposition 9. We list the possible Tits indexes from [**47**] in the first column of Table 10.

4.6.1. *Type E_8 and $p = 2$.* As anisotropic strongly inner groups of types D_4, D_6, D_7, and E_7 exist over 2-special fields by the previous sections, and a compact E_8 exists over $\mathbb{R}$, the "yes" entries in the second column of Table 10 are clear. For the one "no", we refer to Table 4.

4.6.2. *Type E_8 and $p = 3$.* In view of results for groups of smaller rank, it suffices to justify the existence of an anisotropic E_8 over a 3-special field. For this, we refer to Table 11, which is justified in [**22**, §10].

Tits index of G	occurs as a 2-index?	occurs as a 3-index?	occurs as a 5-index?
	yes (8)	yes	yes
	yes (−24)	no	no
	yes	no	no
	no	yes	no
	yes	no	no
	yes	no	no
	yes (−248)	yes	yes

TABLE 10. *Possible Tits indexes for a group of type E_8*

$J_3(G)$	$(0,0)$	$(1,0)$	$(1,1)$
Tits 3-index of G	split		anisotropic
$b(G)$	0	nonzero symbol	otherwise

TABLE 11. *Possible values for the mod-3 J-invariant of G of type E_8, assuming char $k = 0$ and k 3-special*

Flag varieties. Currently there is no concrete description of the flag varieties of E_8 available in the form analogous to the others presented here. However, groups of type E_8 can be viewed as the automorphism group of various algebraic structures as explained in [**19**] (such as a 3875-dimensional algebra, for fields of characteristic $\neq 2$), so the methods of [**6**] can in principle be used to give a concrete description of the flag varieties in terms of such structures.

References

1. M. Aschbacher, *The 27-dimensional module for E_6. I*, Invent. Math. **89** (1987), 159–195.
2. A. Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.
3. A. Borel and T.A. Springer, *Rationality properties of linear algebraic groups. II*, Tôhoku Math. J. (2) **20** (1968), 443–497, (= Borel, Oe., vol. 2, #76).
4. A. Borel and J. Tits, *Compléments à l'article: Groupes réductifs*, Publ. Math. IHES **41** (1972), 253–276.
5. N. Bourbaki, *Lie groups and Lie algebras: Chapters 4–6*, Springer-Verlag, Berlin, 2002.
6. M. Carr and S. Garibaldi, *Geometries, the principle of duality, and algebraic groups*, Expo. Math. **24** (2006), 195–234.
7. V. Chernousov, *A remark on the* (mod 5)*-invariant of Serre for groups of type E_8*, Math. Notes **56** (1995), no. 1-2, 730–733, [Russian original: Mat. Zametki **56** (1994), no. 1, pp. 116-121].
8. C. De Clercq, *Equivalence motivique des groupes semisimples*, preprint, 2015.
9. R.S. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, Colloquium Publications, vol. 56, Amer. Math. Soc., 2008.
10. A.J. Engler and A. Prestel, *Valued fields*, Springer, 2005.

11. H. Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene. VIII, IX*, Nederl. Akad. Wetensch. Proc. Ser. A **62** (1959), 447–474.
12. S. Garibaldi, *Twisted flag varieties of trialitarian groups*, Comm. Alg. **27** (1999), no. 2, 841–856.
13. ———, *Groups of type E_7 over arbitrary fields*, Comm. Algebra **29** (2001), no. 6, 2689–2710.
14. ———, *Structurable algebras and groups of type E_6 and E_7*, J. Algebra **236** (2001), no. 2, 651–691.
15. ———, *Cohomological invariants: exceptional groups and spin groups*, Memoirs Amer. Math. Soc., no. 937, Amer. Math. Soc., Providence, RI, 2009, with an appendix by Detlev W. Hoffmann.
16. ———, *The Rost invariant has trivial kernel for quasi-split trialitarian groups*, arxiv:1004.0663, April 2010.
17. ———, *Outer automorphisms of algebraic groups and determining groups by their maximal tori*, Michigan Math. J. **61** (2012), no. 2, 227–237.
18. S. Garibaldi and P. Gille, *Algebraic groups with few subgroups*, J. London Math. Soc. (2) **80** (2009), 405–430.
19. S. Garibaldi and R.M. Guralnick, *Simple groups stabilizing polynomials*, Forum of Mathematics: Pi **3** (2015), e3 (41 pages).
20. S. Garibaldi, A. Merkurjev, and J-P. Serre, *Cohomological invariants in Galois cohomology*, University Lecture Series, vol. 28, Amer. Math. Soc., 2003.
21. S. Garibaldi and H.P. Petersson, *Groups of outer type E_6 with trivial Tits algebras*, Transf. Groups **12** (2007), no. 3, 443–474.
22. S. Garibaldi, V. Petrov, and N. Semenov, *Shells of twisted flag varieties and the Rost invariant*, Duke Math. J. **165** (2016), no. 2, 285–339.
23. S. Garibaldi and A. Quéguiner-Mathieu, *Pfister's theorem for orthogonal involutions of degree 12*, Proc. Amer. Math. Soc. **137** (2009), 1215–1222.
24. P. Gille, *Invariants cohomologiques de Rost en caractéristique positive*, K-Theory **21** (2000), 57–100.
25. M.-A. Knus, A.S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Colloquium Publications, vol. 44, Amer. Math. Soc., 1998.
26. A. Malagon, *Killing forms of isotropic Lie algebras*, J. Pure Appl. Algebra **216** (2012), no. 10, 2213–2224.
27. W.G. McKay, J. Patera, and D.W. Rand, *Tables of representations of simple Lie algebras. volume I: exceptional simple Lie algebras*, Centre de Recherches Mathématiques, Montreal, 1990.
28. A.S. Merkurjev, *Maximal indexes of Tits algebras*, Doc. Math. J. DMV **1** (1996), 229–243.
29. ———, *Degree three cohomological invariants of semisimple groups*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 2, 657–680.
30. A.S. Merkurjev, I. Panin, and A.R. Wadsworth, *Index reduction formulas for twisted flag varieties, I*, K-Theory **10** (1996), 517–596.
31. ———, *Index reduction formulas for twisted flag varieties, II*, K-Theory **14** (1998), no. 2, 101–196.
32. A.S. Merkurjev and J.-P. Tignol, *The multipliers of similitudes and the Brauer group of homogeneous varieties*, J. reine angew. Math. **461** (1995), 13–47.
33. D. Orlov, A. Vishik, and V. Voevodsky, *An exact sequence for $K_*^M/2$ with applications to quadratic forms*, Annals Math. (2) **165** (2007), no. 1, 1–13.
34. H.P. Petersson, *Structure theorems for Jordan algebras of degree three over fields of arbitrary characteristic*, Comm. Alg. **32** (2004), 1019–1049.
35. ———, *Albert algebras*, available at http://www.fields.utoronto.ca/programs/scientific/11-12/exceptional/Alb.-alg.-Ottawa-2012-Vii-new.pdf, May 2012.
36. H.P. Petersson and M.L. Racine, *Reduced models of Albert algebras*, Math. Z. **223** (1996), no. 3, 367–385.
37. V. Petrov, N. Semenov, and K. Zainoulline, *J-invariant of linear algebraic groups*, Ann. Ec. Norm. Sup. (4) **41** (2008), 1023–1053.
38. V. Petrov and A. Stavrova, *The Tits indices over semilocal rings*, Transform. Groups **16** (2011), no. 1, 193–217.
39. G. Prasad and A.S. Rapinchuk, *On the fields generated by the lengths of closed geodesics in locally symmetric spaces*, Geom. Dedicata **172** (2014), 79–120.
40. M.L. Racine, *Point spaces in exceptional quadratic Jordan algebras*, J. Algebra **46** (1977), 22–36.
41. M. Rost, *On the classification of Albert algebras*, September 2002, unpublished note available at http://www.math.uni-bielefeld.de/~rost/papers.html.
42. O.F.G. Schilling, *The theory of valuations*, Math. Surveys, vol. 4, American Mathematical Society, 1950.
43. J-P. Serre, *Cohomologie galoisienne: progrès et problèmes*, Astérisque (1995), no. 227, 229–257, Séminaire Bourbaki, vol. 1993/94, Exp. 783, (= Oe. 166).
44. T.A. Springer, *Sur les formes quadratiques d'indice zéro*, C.R. Acad. Sci. Paris **234** (1952), 1517–1519.
45. ———, *Linear algebraic groups*, second ed., Birkhäuser, 1998.
46. T.A. Springer and F.D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer-Verlag, Berlin, 2000.
47. J. Tits, *Classification of algebraic semisimple groups*, Algebraic Groups and Discontinuous Subgroups, Proc. Symp. Pure Math., vol. IX, AMS, 1966, pp. 33–62.
48. ———, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. **247** (1971), 196–220.
49. ———, *Sur les degrés des extensions de corps déployant les groupes algébriques simples*, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), no. 11, 1131–1138.
50. A. Weil, *Algebras with involution and the classical groups*, J. Indian Math. Soc. **24** (1960), 589–623.

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