Errata and addenda
for the book
Albert algebras over commutative rings

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Errata for the published edition of *Albert algebras over commutative rings*

Page numbers refer to the published/paper edition of the book, published by Cambridge in November 2024. These typos have been corrected in the online/"draft" version you can find on the web. The online/draft versions have a date on their front cover, and the "fixed" date says that the typo has been corrected in all online/draft versions starting with that date.

This errata sheet was compiled on April 5, 2025.

If you find other typos, please do tell us! You can send us an email or use the contact form on Skip's website at this link.

Notation and conventions

Page xxii, line -17 (fixed 20 Feb 2025): The R in $M \times M \to R$ should be replaced by k.

Chapter I: Prologue: the ancient protagonists

Page 38, item 6.4, line 4 (fixed 16 Dec 2024): The U-operator maps $U\colon J \to \mathrm{End}(J)$. That is, the codomain is $\mathrm{End}(J)$ not J.

Chapter II: Foundations

Page 104, item 12.39, line 2 (fixed 2 Jan 2025): Insert after the first sentence: (See for example [27[1], §VI.3.6] for background on discrete valuations on fields.)

Chapter IV: Composition algebras

Page 157, item 19.24, line 5 (fixed 16 Dec 2024): Replace $\mathrm{Her}_r(\mathbb{D})$ with $\mathrm{Her}_r(\mathbb{O})$.

[1] For the convenience of the reader: [27] in the printed book = [Bou72].

Chapter V: Jordan algebras

Page 275, item 29.11, line 3 (fixed 20 Feb 2025): Replace "that that" with a single "that".

Chapter VII: The two Tits constructions

Page 458, item 42.18, line 1 (fixed 16 Jan 2025): The first sentence should read: Let $J = J(D, \mu)$ be an Albert algebra arising from a first Tits construction over a field F of characteristic $\neq 3$.

Page 509, item 46.6 proof, line 3 (fixed 16 Dec 2024): Should refer to Lemma 46.5, not Proposition.

Chapter VIII: Lie algebras

Page 527, line 1 (fixed 20 Feb 2025): The first sentence should read: Let V be the subspace of $E = \mathbb{R}^8$ whose points have coordinates (ξ_i) such that $\xi_6 = \xi_7 = -\xi_8$.

(That is, replace every ϵ in the printed version with a ξ, to align with the notation in [Bou02].)

Page 529, item 47.17, line 4 of the remark (fixed 20 Feb 2025): Replace "very" with "every".

Page 565, 51.29 (fixed 20 Feb 2025): The statement of part (a) should also include the hypothesis $2 \in k^\times$, in order for the proof provided to be sufficient. The proof needs that the map $\mathfrak{g}_0 \to \mathfrak{o}(n_C)$ is bijective from 51.26(b), which relies on $2 \in k^\times$.

The claim in part (a), that $\mathrm{Der}(J)$ is finitely generated projective of rank 52, does hold in greater generality. Here is a proof using the material from the next chapter, Chapter IX.

First suppose that k is a field, in which case we only need to prove that $\dim(\mathrm{Der}(J)) = 52$. By 52.1(b), $\mathrm{Lie}(\mathbf{Aut}(J)) = \mathrm{Der}(J)$. By Theorem 53.4, $\mathbf{Aut}(J)$ is semisimple of type F_4, so $\dim \mathrm{Lie}(\mathbf{Aut}(J)) = \dim \mathbf{Aut}(J)$ because $\mathbf{Aut}(J)$ is smooth and $\dim \mathbf{Aut}(J) = 52$ because type F_4.

Next suppose that J is the split Albert algebra $\mathrm{Her}_3(\mathrm{Zor}(k))$ over a principal ideal domain k. Since $\mathrm{Zor}(k)$ is a finitely generated free module, so is $\mathrm{End}(\mathrm{Zor}(k))$; since additionally k is a principal ideal domain we conclude that the submodule $\mathrm{Der}(J)$ is also finitely generated and free [Sta18, Tag 0AUW]. The field of fractions F of k is a flat extension of k, so $\mathrm{Der}(J)_F \cong \mathrm{Der}(J_F)$ by Prop. 50.4, and this has dimension 52 by the previous paragraph, hence $\mathrm{Der}(J)$ has rank 52 at the prime 0. Since k is connected, $\mathrm{Der}(J)$ has constant rank 52.

If J is any Albert algebra over a principal ideal domain k, then there is some faithfully flat $K \in k$-**alg** such that J_K is split. Then $\mathrm{Der}(J)_K \cong \mathrm{Der}(J_K) \cong \mathrm{Der}(\mathrm{Her}_3(\mathrm{Zor}(K))) \cong \mathrm{Der}(\mathrm{Her}_3(\mathrm{Zor}(k))) \otimes K$ is finitely generated projective of rank 52 by the previous paragraph, and it follows that $\mathrm{Der}(J)$ is finitely generated projective of rank 52.

Page 557, item (51.10.2), line 1 (fixed 16 Dec 2024): The right side of the equation should read $\{(a_1, a_1^\top) \mid a_1 \in \mathrm{Mat}_3(k)\}$.

Page 566, alternative proof of 51.29(b), line 2 (fixed 20 Feb 2025): Replace $\mathrm{Zor}(k)$ with $\mathrm{Her}_3(\mathrm{Zor}(k))$.

Chapter IX: Group schemes

Page 567, item 52.1, paragraph 2, line -2 (fixed 22 Feb 2025): Replace G with $\mathbf{G}$.

Page 588, item proof of 54.6, line 3 (fixed 20 Feb 2025): Replace “(1 3 2)” by “(1 2)”.

Page 593, item 54.11, line 2 (fixed 20 Feb 2025): Replace “structure” by “system”.

Page 593, item 54.12, line 2 (fixed 20 Feb 2025): Replace “structure” by “system”.

Page 603, 55.12 (fixed 29 Mar 2025): Replace the statement of the exercise with the following, clearer, version:

Let F be a field. Show that the maps $(A, \tau) \mapsto H(A, \tau)$ and $J \mapsto \mathbf{Aut}(J)$ define bijections between the isomorphism classes of

(i) Azumaya algebras (A, τ) of degree 3 with unitary involution over F, as in 44.23,
(ii) rank 9 Freudenthal F-algebras J, and
(iii) adjoint semi-simple F-group schemes of type A_2.

Page 619, bottom of page (fixed 20 Feb 2025): Insert new paragraph: Results for groups of type E_7 — analogous to the results in this section for type E_6 — can be found in sections 16 and 17 of [95[2]].

[2] For the convenience of the reader: [95] in the printed book = [GPR23].

Addenda to the published edition of *Albert algebras over commutative rings*

This chapter contains material that could have been in the book and does not require substantial extra background.

A1 Albert algebras are exceptional

In his first paper on the subject of Jordan algebras, Albert [Alb34] showed that (1) the euclidean Albert algebra $\mathrm{Her}_3(\mathbb{O})$ over the reals is a (linear) Jordan algebra and (2) that it is exceptional. We have shown (1) in Theorem 5.10. More generally, for every octonion algebra C over every ring k, we have shown that $\mathrm{Her}_3(C)$ is a Jordan algebra in Theorem 36.5. (Note that Albert algebras are defined in the book to be cubic Jordan algebras, hence Jordan algebras.) In this addendum we would like to extend (2), the exceptionality of Albert algebras, to an arbitrary ring k. We prove the following slightly stronger result.

A1.1 Theorem. *Every Albert algebra over every non-zero ring is i-exceptional.*

We will define the term i-exceptional in a moment; it is stronger than the notion of exceptional defined in 29.9. When the ring is a field of characteristic different from 2, Theorem A1.1 was first proved in [AP59]. A proof of Theorem A1.1 for every field can be found in Jacobson's Arkansas notes [Jac81, §2.5]. That proof considers separately the cases of fields of characteristic 2 and different from 2. We provide a proof that does not contain special considerations involving 2.

In the excluded case of the zero ring, all algebras are zero and so the notion of being exceptional or i-exceptional do not make sense.

Here is a beautiful application of the theorem.

A1.2 Theorem. *A central simple Jordan algebra over a field is exceptional if and only if it is an Albert algebra.*

Proof The central simple exceptional Jordan algebras over a field have been classified into types in §15 of [MZ88]. One finds that each type is either special or an Albert algebra. Albert algebras are exceptional by Theorem A1.1. □

We now provide the promised definition.

A1.3 Definition. A Jordan algebra J is *i-special* if J is a homomorphic image of a special algebra and *i-exceptional* if it is not.

Clearly any special Jordan algebra is i-special. However P.M. Cohn has given an example of an i-special Jordan algebra that is exceptional [Jac68, §I.3, Thm. 2]. Therefore being i-exceptional is stronger than being exceptional. We also have the following:

A1.4 Lemma. *Let J be a Jordan k-algebra. If J_R is i-exceptional for some flat $R \in k$-**alg**, then J is i-exceptional.*

Proof We prove the contrapositive. Suppose that J is not i-exceptional, i.e., there is a special Jordan algebra J' and a (surjective) homomorphism $f\colon J' \to J$. Hence there exists a unital associative algebra A and an injective Jordan homomorphism $j\colon J' \to A^{(+)}$. It follows that $f_R\colon J'_R \to J_R$ is still surjective and, by flatness, $j_R\colon J'_R \to (A_R)^{(+)}$ is still injective, i.e., J_R is i-special. □

As a first step in the proof of Theorem A1.1, we note the following, which could have appeared in 37.7.

A1.5 Lemma. *For $J = \mathrm{Her}_3(C)$, C a multiplicative conic alternative algebra, and $a[ij]$, $b[ji]$, $c[il] \in J$ with $\{i, j, k\} = \{1, 2, 3\}$,*

$$\{a[ij]\, b[ji]\, c[il]\} = a(bc)[il]. \tag{a1}$$

We remark that if C is associative, then (a1) follows immediately from matrix multiplication.

Proof Using (33.6.2), (36.4.6), (36.4.7),

$$\begin{aligned}\{a[ij]\, b[ji]\, c[il]\} &= T_J(a[ij], \bar{b}[ij])c[il] - (c[il] \times a[ij]) \times b[ji] \\ &= n_C(a, \bar{b})c[il] - \bar{a}c[jl] \times b[ji] \\ &= (n_C(a, \bar{b})c - \bar{b}(\bar{a}c))[il].\end{aligned}$$

Replace $\bar{a}$ and $\bar{b}$ using (16.5.4) and expand. By (16.5.5), Proposition 16.10 and alternativity, we obtain (a1). □

The following polynomial map was introduced by Glennie [Gle66]:

$$g_9(x,y,z) := U_x z \circ \{x\, U_z y^2\, y\} - U_y z \circ \{x\, U_z x^2\, y\} \\ - U_x U_z\{x\, U_y z\, y\} + U_y U_z \{x\, U_x z\, y\}.$$

It is homogeneous of degree 9 and skew-symmetric in x and y.

A1.6 Lemma. *Let C be a multiplicative conic alternative algebra. If C is not associative, then g_9 is not identically zero on* $\mathrm{Her}_3(C)$.

Proof Consider the following substitution,

$$x = 1[12], \quad y = 1[23], \quad z = a[12] + b[23] + c[31], \quad x, y, z \in \mathrm{Her}_3(C),$$

for a, b, $c \in C$. Using (16.12.2), (17.4.2), the Peirce decomposition rules (32.15), the identities of 37.7 and Lemma A1.5, one computes

$$x^2 = e_{11} + e_{22}, \quad y^2 = e_{22} + e_{33}, \quad U_x z = \bar{a}[12], \quad U_y z = \bar{b}[23].$$

Futhermore

$$\begin{aligned}
U_z x^2 &= n_C(a)e_{11} + n_C(a)e_{22} + (n_C(b) + n_C(c))e_{33} + \bar{a}\,\bar{c}[23] + \bar{b}\,\bar{a}[31].\\
U_z y^2 &= n_C(b)e_{22} + n_C(b)e_{33} + (n_C(a) + n_C(c))e_{11} + \bar{c}\,\bar{b}[12] + \bar{b}\,\bar{a}[31].\\
\{x\, U_z y^2\, y\} &= t_C(ab)e_{22} + \bar{c}\,\bar{b}[23] + n_C(b)1[31].\\
\{x\, U_z x^2\, y\} &= t_C(ab)e_{22} + \bar{a}\,\bar{c}[12] + n_C(a)1[31].\\
U_x z \circ \{x\, U_z y^2\, y\} &= t_C(ab)\bar{a}[12] + n_C(b)a[23] + (bc)a[31].\\
U_y z \circ \{x\, U_z x^2\, y\} &= n_C(a)b[12] + t_C(ab)\bar{b}[23] + b(ca)[31].\\
\{x\, U_y z\, y\} &= \bar{b}[12],\\
U_z\{x\, U_y z\, y\} &= n_C(b)t_C(c)e_{33} + aba[12] + n_C(b)\bar{a}[23] + (c\bar{b})\bar{a}[31].\\
U_x U_z\{x\, U_y z\, y\} &= \overline{aba}[12].\\
\{x\, U_x z\, y\} &= \bar{a}[23].\\
U_z\{x\, U_x z\, y\} &= n_C(a)t_C(c)e_{11} + n_C(a)\bar{b}[12] + bab[23] + \bar{b}(\bar{a}c)[31].\\
U_y U_z\{x\, U_x z\, y\} &= \overline{bab}[23].
\end{aligned}$$

Therefore

$$g_3(x,y,z) = (t_C(ab)\bar{a} - n_C(a)b - \overline{aba})[12] + (n_C(b)a - t_C(ab)\bar{b} + \overline{bab})[23] \\ + [b,c,a][31].$$

The [12] and [23] entries are 0 by (17.4.2). If C is not associative, one can choose $a, b, c \in C$ such that the associator $[a,b,c] \neq 0$, in which case the [31] entry is not 0. ◻

A1.7 Lemma. *The Glennie polynomial $g_9(x, y, z)$ vanishes on every i-special Jordan algebra.*

Proof A special Jordan algebra is a subalgebra of $A^{(+)}$ for some associative algebra A. Let u, v, $w \in A$. Substituting $x = u$, $y = v$, $z = w$ in $g_9(x, y, z)$, we obtain

$$\begin{aligned} uwuuwv^2wv + uwuvwv^2wu + uwv^2wvuwu + vwv^2wuuwu \\ - vwvuwu^2wv - vwvvwu^2wu - uwu^2wvvwv - vwu^2wuvwv \\ - uwuvwvvwu - uwvvwvuwu + vwuuwuvwv + vwvuwuuwv = 0. \end{aligned}$$

If J is an i-special Jordan algebra, then by definition there is a surjection $f\colon J' \to J$ such that J' is special. Then g_9 on J can be computed as the composition $g_9 f$ on J', which is identically zero by the preceding paragraph. □

In fancier language, we have shown that g_9 is an *s-identity*, i.e., it is not identically zero for every Jordan algebra (Lemma A1.6) and it vanishes on every special Jordan algebra (Lemma A1.7).

Proof of Theorem A1.1 If J is an Albert algebra, by Corollary 39.32, there exists a faithfully flat extension $R \in k$-**alg** such that $J_R \cong \mathrm{Her}_3(\mathrm{Zor}(R))$. Since $\mathrm{Zor}(R)$ is not associative, Lemmas A1.6 and A1.7 show that J_R is not i-special, i.e., is i-exceptional, and it follows (Lemma A1.4) that J is i-exceptional. □

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